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Asymptotic properties of solutions to linear non-autonomous neutral differential equations

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Abstract

In this article we study asymptotic properties of solutions to first order linear neutral differential equations with variable coefficients and constant delays. Results are stated in terms of the solution to a characteristic equation. By doing this, we extend some of the results obtained for delay equations in [J.G. Dix, Ch.G. Philos, I.K. Purnaras, An asymptotic property of solutions to linear non-autonomous delay differential equations, Electron. J. Differential Equations 2005 (2005) 1–9] to neutral equations.

Keywords: Neutral differential equation; Asymptotic behavior; Characteristic equation

1. Introduction

We study asymptotic properties of solutions to the neutral differential equation

$$x'(t) + \sum_{i\in I} c_i(t)x'(t-\sigma_i) = a(t)x(t) + \sum_{j\in J} b_j(t)x(t-\tau_j), \quad \text{for } t \geqslant 0 \tag{1.1}$$

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with initial condition

$$x(t)=\phi(t), \quad \text{for } \min\{-\sigma_i, -\tau_j\colon\ i \in I,\ j \in J\} \leqslant t \leqslant 0, \tag{1.2}$$

where I and J are initial segments of natural numbers, the function coefficients a, b_j, c_i are continuous on $[0,\infty)$, the delays σ_i, τ_j are positive real numbers, and ϕ is a given differentiable function.

Asymptotic behavior of solutions and their stability has been studied using several methods; among them we have Lyapunov functionals (or energy functionals), fixed points (or contraction mappings), and characteristic equations. Each of these methods has its own advantages and disadvantages when studying retarded (delay), advanced, and neutral equations. Lyapunov functionals seem to work well for delay equations and have been used in publications such as [6,20]. However, Lyapunov functionals are not available for certain equations. This is the case in the present paper. Fixed points and, in particular, contraction mappings have been used in articles such as [2–4,12,20]. Usually contraction mappings lead to weaker conditions for stability than Lyapunov functionals and characteristic equations. However, they are not always available. Meanwhile characteristic equations work well for constant delays in advanced and neutral equations. They have been used in publications such as [1,7–11,13,16–19]. However the conditions obtained in [7] on the coefficients seem to be very restrictive.

In the present paper, we extend some of the results obtained for delay differential equations in [7] to neutral differential equations. Statements about solutions of (1.1) are made in terms of solutions to a characteristic equation. Our main result is formulated as Theorem 2.4 in the next section and then proved in Section 3.

In general, the theory of neutral differential equations presents complications, and results which are true for non-neutral equations may not be true for neutral equations. Besides its theoretical interest, the study of the asymptotic behavior of solutions of neutral differential equations has some importance in applications. Neutral differential equations appear in networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar and also as the Euler equation in some variational problems. For the basic theory of neutral differential equations, we refer the reader to the books [5,14,15].

2. Statement of results

We shall assume that the delays are positive and denote

$$\sigma=\max\{\sigma_i\colon\ i\in I\}, \qquad \tau=\max\{\tau_j\colon\ j\in J\}, \qquad r=\max\{\sigma,\tau\}.$$

Let $C([-r,0],\mathbb{R})$ denote the set of continuous real-valued functions on $[-r,0]$.

By a solution x to the neutral differential equation (1.1), we mean a continuous real-valued function defined on $[-r,\infty)$ and continuously differentiable on $[-\sigma,\infty)$ and that satisfies (1.1).

With the neutral equation (1.1), we associate the integral equation

$$\lambda(t)+\sum_{i\in I} c_i(t)\lambda(t-\sigma_i)\exp\left[-\int_{t-\sigma_i}^{t}\lambda(s)\,ds\right]$$

$$= a(t) + \sum_{j\in J} b_j(t) \exp\left[-\int_{t-\tau_j}^{t} \lambda(s)\,ds\right], \quad \text{for } t \geqslant 0;$$

$$\lambda(t) = \lambda_0(t), \quad \text{for } -r \leqslant t \leqslant 0 \tag{2.1}$$

which is called the (generalized) *characteristic equation* of (1.1). This equation is obtained when looking for solutions of the form

$$x(t) = \phi(0) \exp\left[\int_0^t \lambda(s)\,ds\right], \quad t \geqslant -r.$$

Note that this solution cannot change sign; i.e., $x(t)$ is either positive or negative or identically zero.

By a solution λ to the characteristic equation, we mean a continuous real-valued function, defined on $[-r, \infty)$, which satisfies (2.1).

Lemma 2.1. *For each λ_0 in $C([-r,0],\mathbb{R})$, the characteristic equation has a unique global solution.*

Proof. Let $r_1 = \min\{\sigma_i, \tau_j\colon\ i \in I,\ j \in J\}$ and recall that $r = \max\{\sigma_i, \tau_j\colon\ i \in I,\ j \in J\}$. Let $u(t) = \exp[\int_0^t \lambda(s)\,ds]$ for $t \geqslant -r$. Recall that $\lambda(t) = \lambda_0(t)$ for $-r \leqslant t \leqslant 0$. Let $w(t) = \exp[\int_0^t \lambda(s)\,ds]$ for $-r \leqslant t \leqslant 0$. Then using the characteristic equation, for $0 \leqslant t \leqslant r_1$, we obtain the linear differential equation

$$u'(t) = \lambda(t)u(t) = a(t)u(t) - \sum_{i\in I} c_i(t)\lambda(t-\sigma_i)w(t-\sigma_i) + \sum_{j\in J} b_j(t)w(t-\tau_j)$$

with $u(0) = 1$. The solution to this equation is

$$u(t) = \left\{1 + \int_0^t \left(-\sum_{i\in I} c_i(s)\lambda_0(s-\sigma_i) \exp\left[\int_0^{s-\sigma_i} \lambda_0(v)\,dv\right]\right.\right.$$
$$\left.\left. + \sum_{j\in J} b_j(s) \exp\left[\int_0^{s-\tau_j} \lambda_0(v)\,dv\right]\right) \exp\left[-\int_0^s a(v)\,dv\right] ds\right\} \exp\left[\int_0^t a(v)\,dv\right]$$

which allows defining $\lambda(t) = u'(t)/u(t)$ on $[0, r_1]$. For the next step, let $w(t) = u(t)$ on $[-r, r_1]$. Then for $r_1 \leqslant t \leqslant 2r_1$, we obtain the differential equation

$$u'(t) = \lambda(t)u(t) = a(t)u(t) - \sum_{i\in I} c_i(t)\lambda(t-\sigma_i)w(t-\sigma_i) + \sum_{j\in J} b_j(t)w(t-\tau_j)$$

whose solution allows us defining $\lambda(t)$ on $[r_1, 2r_1]$. Proceeding in this manner, we define $\lambda(t)$ for all $t \geqslant -r$, which completes the proof. $\square$

Once a solution to the characteristic equation is available, we can build solutions to the neutral equation.

Lemma 2.2. *For each initial function ϕ continuously differentiable on* $[-r, 0]$*, there exists a unique solution to* (1.1), (1.2).

Proof. When $\phi(t)$ does not have zeros on $[-r, 0]$, we define λ_0 with the expression $\phi(t) = \phi(0)\exp[\int_0^t \lambda_0(s)\,ds]$; i.e., $\lambda_0(t) = \phi'(t)/\phi(t)$. By Lemma 2.1, for this initial function, the characteristic equation has solution that defines $x(t) = \phi(0)\exp[\int_0^t \lambda_0(s)\,ds]$ as a solution of (1.1), (1.2) on $[-r, \infty)$. In particular, (1.1) with $\phi(t) = k$ a non-zero constant has a solution that we denote by x_k.

Now, we consider the case when ϕ has zeros on $[-r, 0]$. Since ϕ is continuous on the closed and bounded interval $[-r, 0]$, there is a non-zero constant k such that $\psi(t) := \phi(t) + k > 0$ on $[-r, 0]$. Then (1.1) with initial function ψ has a solution y. Since (1.1) is a linear problem, $x(t) = y(t) - x_k(t)$ is a solution of (1.1), (1.2).

The uniqueness of solutions can be obtained as follows: If x and y are solutions of (1.1), (1.2), then $w(t) = x(t) - y(t)$ is a solution of (1.1) with zero initial function. Using the method of steps, $w(t) \equiv 0$ on $[-r, 0]$ implies $w(t) \equiv 0$ on $[0, r_1]$, where $r_1 = \min\{\sigma_i, \tau_j\colon\ i \in I,\ j \in J\}$. Repeating this process, $w(t) \equiv 0$ on $[r_1, 2r_1]$, and so on. Therefore, $x = y$. $\square$

Remark 2.3. If the solution to (1.1), (1.2) does not have zeros on some interval $[t^* - r, t^*]$, then the solution does not have zeros on $[t^*, \infty)$; i.e., the solution cannot change sign on $[t^*, \infty)$. To show this claim let t^* be the initial time for the characteristic equation and $\lambda_0(t)$ be given implicitly by $x(t) = x(t^*)\exp[\int_{t^*}^t \lambda_0(s)\,ds]$, with $t^* - r \leqslant t \leqslant t^*$. Then, by the uniqueness of solutions to (1.1),

$$x(t) = x(t^*)\exp\left[\int_{t^*}^{t} \lambda(s)\,ds\right], \quad \text{for } t \geqslant t^*.$$

Therefore, $x(t)$ cannot have zeros on $[t^*, \infty)$.

Our main result is the following theorem.

Theorem 2.4. *Assume that*

$$\sup_{t \geqslant r-\sigma}\left\{\sum_{i\in I}\left|c_i(t)\right|\left[1 + \sigma_i\left|\lambda(t-\sigma_i)\right|\right]\exp\left[-\int_{t-\sigma_i}^{t} \lambda(s)\,ds\right] + \sum_{j\in J}\left|b_j(t)\right|\tau_j\exp\left[-\int_{t-\tau_j}^{t} \lambda(s)\,ds\right]\right\} < 1. \tag{2.2}$$

Then for each solution x of (1.1), (1.2) *there exists a constant L_{ϕ,λ_0} such that*

$$\lim_{t\to\infty} x(t)\exp\left[-\int_0^t \lambda(s)\,ds\right] = L_{\phi,\lambda_0}$$

and

$$\lim_{t\to\infty}\left\{x(t)\exp\left[-\int_0^t \lambda(s)\,ds\right]\right\}'=0.$$

Remark 2.5. Under the conditions of Theorem 2.4, a solution to (1.1) cannot grow faster than the exponential function determined by the characteristic equation; i.e., there exists a constant M such that

$$\left|x(t)\right| \leqslant M\exp\left[\int_0^t \lambda(s)\,ds\right], \quad \text{for } t \geqslant 0.$$

Remark 2.6. When the solution to (2.1) is a constant λ_0 satisfying (2.2),

$$\lim_{t\to\infty} x(t)\exp(-t\lambda_0)=L_{\phi,\lambda_0}.$$

In particular, when zero is the solution to (2.1), $\lim_{t\to\infty} x(t)=L_{\phi,0}$.

Note that if λ is a solution of (2.1), then

$$x(t)=\phi(0)\exp\left[\int_0^t \lambda(s)\,ds\right]$$

is a solution of (1.1) with initial function $\phi(t)=\phi(0)\exp[\int_0^t \lambda(s)\,ds]$. Then we obtain easily the following results.

Remark 2.7. Under the assumptions of Theorem 2.4, we have:

(1) Every solution of (1.1) is bounded if and only if $\limsup_{t\to\infty}\int_0^t \lambda(s)\,ds<\infty$.
(2) Every solution of (1.1) tends to zero at ∞ if and only if $\lim_{t\to\infty}\int_0^t \lambda(s)\,ds=-\infty$.

We remark that when $c_i=0$ for all $i\in I$, Eq. (1.1) is the equation studied in [7]. However, we were unable to prove a result equivalent to [7, Lemma 4.2] which states conditions on the coefficients of (1.1) to have condition (2.2).

Also, it is remarkable that in the special case of first order linear autonomous neutral differential equations, the (generalized) characteristic equation coincides with the usual characteristic equation. Moreover, it must be noted that in the particular case of a class of first order linear periodic neutral differential equations (such as treated in [18]), an equation is introduced which is also called the characteristic equation; this equation can be used instead of the (generalized) characteristic equation. In both cases, an explicit expression (in terms of ϕ and λ_0) can be obtained for the real number L_{ϕ,λ_0} in Theorem 2.4. See [11,16,18,19] (see, also, [8,10,13,17] for the special case of non-neutral equations). This is achieved by the utilization of an integral representation of a function y associated with the solution x of the differential equation. In the non-autonomous case (considered in the present paper as well as in [7] for delay equations), we have no such a representation of the function y itself; but we have an integral representation of the derivative of y, which can be utilized to prove Theorem 2.4.

3. Proof of main result

Proof of Theorem 2.4. For solutions x of (1.1), (1.2) and λ of (2.1), we define the function

$$y(t) = x(t)\exp\left[-\int_0^t \lambda(s)\,ds\right], \quad t \geqslant -r.$$

We shall prove that $\lim_{t\to\infty} y(t)$ exists and that $\lim_{t\to\infty} y'(t) = 0$. Differentiating y, and using (1.1), (2.1), we obtain

$$\begin{aligned} y'(t) &= \big(x'(t) - x(t)\lambda(t)\big)\exp\left[-\int_0^t \lambda(s)\,ds\right] \\ &= \Bigg(-\sum_{i\in I} c_i(t)x'(t-\sigma_i) + x(t)\sum_{i\in I} c_i(t)\lambda(t-\sigma_i)\exp\left[-\int_{t-\sigma_i}^t \lambda(s)\,ds\right] \\ &\quad + \sum_{j\in J} b_j(t)x(t-\tau_j) - x(t)\sum_{j\in J} b_j(t)\exp\left[-\int_{t-\tau_j}^t \lambda(s)\,ds\right]\Bigg) \\ &\quad \times \exp\left[-\int_0^t \lambda(s)\,ds\right]. \end{aligned}$$

Using that $x(t-\tau_j) = y(t-\tau_j)\exp[\int_0^{t-\tau_j}\lambda(s)\,ds]$, and $x'(t) = (y'(t) + y(t)\lambda(t)) \times \exp[\int_0^t \lambda(s)\,ds]$, the above equality yields

$$\begin{aligned} y'(t) = &-\sum_{i\in I} c_i(t)\big[y'(t-\sigma_i) - \big(y(t) - y(t-\sigma_i)\big)\lambda(t-\sigma_i)\big]\exp\left[-\int_{t-\sigma_i}^t \lambda(s)\,ds\right] \\ &- \sum_{j\in J} b_j(t)\big[y(t) - y(t-\tau_j)\big]\exp\left[-\int_{t-\tau_j}^t \lambda(s)\,ds\right] \quad \text{for } t \geqslant 0. \end{aligned}$$

From this equation and the fundamental theorem of calculus,

$$\begin{aligned} y'(t) = &-\sum_{i\in I} c_i(t)\left[y'(t-\sigma_i) - \lambda(t-\sigma_i)\int_{t-\sigma_i}^t y'(s)\,ds\right]\exp\left[-\int_{t-\sigma_i}^t \lambda(s)\,ds\right] \\ &- \sum_{j\in J} b_j(t)\left[\int_{t-\tau_j}^t y'(s)\,ds\right]\exp\left[-\int_{t-\tau_j}^t \lambda(s)\,ds\right] \quad \text{for } t \geqslant r-\sigma. \end{aligned} \tag{3.1}$$

If all b_j's and c_i's are identically zero on $[r-\sigma,\infty)$, from (3.1), $y'=0$ and y is constant on $[r-\sigma,\infty)$ which would complete the proof. Therefore, we assume that at least one b_j or one c_i is not identically zero on $[r-\sigma,\infty)$. Let

$$\mu_{\lambda_0}=\sup_{t\geqslant r-\sigma}\left\{\sum_{i\in I}\left|c_i(t)\right|\left[1+\sigma_i\left|\lambda(t-\sigma_i)\right|\right]\exp\left[-\int_{t-\sigma_i}^{t}\lambda(s)\,ds\right]\right.$$
$$\left.+\sum_{j\in J}\left|b_j(t)\right|\tau_j\exp\left[-\int_{t-\tau_j}^{t}\lambda(s)\,ds\right]\right\}.$$

Then, by (2.2),

$$0<\mu_{\lambda_0}<1. \tag{3.2}$$

Note that the maximum of $|y'|$ on $[-\sigma,r-\sigma]$ depends on x and λ; hence, on the initial functions ϕ and λ_0. Let

$$M_{\phi,\lambda_0}=\max\left\{\left|y'(t)\right|:\ -\sigma\leqslant t\leqslant r-\sigma\right\}.$$

We shall show that M_{ϕ,λ_0} is also a bound of $|y'|$ on the whole interval $[-\sigma,\infty)$; i.e.,

$$\left|y'(t)\right|\leqslant M_{\phi,\lambda_0}\quad\text{for all } t\geqslant-\sigma. \tag{3.3}$$

On the contrary, assume that there exist $\epsilon>0$ and $t\geqslant-\sigma$ such that $|y'(t)|>M_{\phi,\lambda_0}+\epsilon$. Since $|y'(t)|\leqslant M_{\phi,\lambda_0}$ for $-\sigma\leqslant t\leqslant r-\sigma$, by the continuity of y', there exists $t^*>r-\sigma$ such that

$$\left|y'(t)\right|<M_{\phi,\lambda_0}+\epsilon,\quad \text{for } -\sigma\leqslant t<t^*,\quad\text{and}\quad \left|y'\left(t^*\right)\right|=M_{\phi,\lambda_0}+\epsilon.$$

Using the definition of μ_{λ_0}, (3.1) and (3.2), we obtain

$$\begin{aligned}
M_{\phi,\lambda_0}+\epsilon&=\left|y'\left(t^*\right)\right|\\
&\leqslant\sum_{i\in I}\left|c_i\left(t^*\right)\right|\left[\left|y'\left(t^*-\sigma_i\right)\right|+\left|\lambda\left(t^*-\sigma_i\right)\right|\int_{t^*-\sigma_i}^{t^*}\left|y'(s)\right|ds\right]\\
&\quad\times\exp\left[-\int_{t^*-\sigma_i}^{t^*}\lambda(s)\,ds\right]\\
&\quad+\sum_{j\in J}\left|b_j\left(t^*\right)\right|\left[\int_{t^*-\tau_j}^{t^*}\left|y'(s)\right|ds\right]\exp\left[-\int_{t^*-\tau_j}^{t^*}\lambda(s)\,ds\right]\\
&\leqslant\left(M_{\phi,\lambda_0}+\epsilon\right)\left\{\sum_{i\in I}\left|c_i\left(t^*\right)\right|\left[1+\sigma_i\left|\lambda\left(t^*-\sigma_i\right)\right|\right]\exp\left[-\int_{t^*-\sigma_i}^{t^*}\lambda(s)\,ds\right]\right.
\end{aligned}$$

$$+\sum_{j\in J}\left|b_j\left(t^*\right)\right|\tau_j \exp\left[-\int_{t^*-\tau_j}^{t^*}\lambda(s)\,ds\right]\Bigg\}$$
$$\leqslant (M_{\phi,\lambda_0}+\epsilon)(\mu_{\lambda_0}) < M_{\phi,\lambda_0}+\epsilon,$$

which is a contradiction. Therefore, inequality (3.3) holds. If $M_{\phi,\lambda_0}=0$, from (3.3) it follows that $y'=0$ and y is constant on $[-\sigma,\infty)$, which would complete the proof. Therefore, we assume that $M_{\phi,\lambda_0}>0$.

In view of (3.1) and (3.3),

$$\begin{aligned}\left|y'(t)\right| &\leqslant \sum_{i\in I}\left|c_i(t)\right|\left[\left|y'(t-\sigma_i)\right|+\left|\lambda(t-\sigma_i)\right|\int_{t-\sigma_i}^{t}\left|y'(s)\right|ds\right]\\ &\quad\times\exp\left[-\int_{t-\sigma_i}^{t}\lambda(s)\,ds\right]\\ &\quad+\sum_{j\in J}\left|b_j(t)\right|\left[\int_{t-\tau_j}^{t}\left|y'(s)\right|ds\right]\exp\left[-\int_{t-\tau_j}^{t}\lambda(s)\,ds\right]\\ &\leqslant M_{\phi,\lambda_0}\Bigg\{\sum_{i\in I}\left|c_i(t)\right|\left[1+\sigma_i\left|\lambda(t-\sigma_i)\right|\right]\exp\left[-\int_{t-\sigma_i}^{t}\lambda(s)\,ds\right]\\ &\quad+\sum_{j\in J}\left|b_j(t)\right|\tau_j\exp\left[-\int_{t-\tau_j}^{t}\lambda(s)\,ds\right]\Bigg\}\\ &\leqslant M_{\phi,\lambda_0}(\mu_{\lambda_0}),\quad \text{for } t\geqslant r-\sigma.\end{aligned}$$

Using this inequality, we can show by induction that

$$\left|y'(t)\right|\leqslant M_{\phi,\lambda_0}(\mu_{\lambda_0})^n \quad \text{for } t\geqslant nr-\sigma\ (n=0,1,\ldots). \tag{3.4}$$

For an arbitrary $t\geqslant-\sigma$, we set $n=\lfloor(t+\sigma)/r\rfloor$ (the greatest integer less than or equal to $(t+\sigma)/r$). Then $t\geqslant nr-\sigma$ and $\frac{t+\sigma}{r}-1<n$. Thus, by (3.2) and (3.4),

$$\left|y'(t)\right|\leqslant M_{\phi,\lambda_0}(\mu_{\lambda_0})^n\leqslant M_{\phi,\lambda_0}(\mu_{\lambda_0})^{\frac{t+\sigma}{r}-1}. \tag{3.5}$$

As $t\to\infty$, we have $n\to\infty$ and, by (3.2), $(\mu_{\lambda_0})^n\to 0$. Therefore, by (3.5),

$$\lim_{t\to\infty} y'(t)=0,$$

which proves the second limit in Theorem 2.4.

To prove that $\lim_{t\to\infty} y(t)$ exists (as a real number), we use the Cauchy convergence criterion. For $t>T\geqslant-\sigma$, from (3.5), we have

$$\left|y(t)-y(T)\right|\leqslant\int_T^t\left|y'(s)\right|ds\leqslant\int_T^t M_{\phi,\lambda_0}(\mu_{\lambda_0})^{\frac{s+\sigma}{r}-1}\,ds$$

$$= M_{\phi,\lambda_0} \frac{r}{\ln(\mu_{\lambda_0})} \left[(\mu_{\lambda_0})^{\frac{s+\sigma}{r}-1} \right]_{s=T}^{s=t}$$
$$= M_{\phi,\lambda_0} \frac{r}{\ln(\mu_{\lambda_0})} \left[(\mu_{\lambda_0})^{\frac{t+\sigma}{r}-1} - (\mu_{\lambda_0})^{\frac{T+\sigma}{r}-1} \right].$$

As $T \to \infty$, we have $t \to \infty$ and, by (3.2), the two right-most terms above approach zero. Therefore, $\lim_{T\to\infty} |y(t) - y(T)| = 0$ which by the Cauchy convergence criterion implies the existence of $\lim_{t\to\infty} y(t)$. We call this limit L_{ϕ,λ_0} because it depends on y which in turn depends on the initial functions ϕ and λ_0. This shows the first limit in Theorem 2.4 and completes the proof. □

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